Modern approaches to quantum gravity

Homework 11 — Fall 2025

1. Scalar field in AdS

Consider Minkowskian AdS_{d+1} in conformal coordinates

$$ds^2 = \frac{1}{\cos^2\theta}(-dt^2 + d\theta^2 + \sin^2\theta d\Omega_{d-1}^2) \tag{1}$$

We will find the spectrum of a scalar particle whose field obeys the Klein-Gordon equation

$$\Box\phi = m^2\phi \tag{2}$$

where $\Box$ is taken in AdS_{d+1}.

(a) Argue, from the isometries of AdS_{d+1}, that a general solution of the above equation is given by a linear combination of the solutions

$$\phi = e^{-i\omega t}G(\theta)Y_\ell(\Omega) \tag{3}$$

where Y_ℓ are the spherical Harmonics on the $d-1$ sphere S^{d-1}, with coordinates Ω, and $G(\theta)$ is for now an arbitrary function of θ.

(b) Show that with the above Ansatz, the Klein-Gordon equation reduces to

$$\omega^2\cos^2\theta G(\theta) + \frac{d-1}{\tan\theta}G'(\theta) + \cos^2\theta G''(\theta) - \frac{\ell(\ell+d-2)}{\tan^2\theta}G(\theta) - m^2G(\theta) = 0 \tag{4}$$

(c) Show that the above equation are solved by the solutions $G_\pm$, where

$$G_+ = (\sin\theta)^\ell(\cos\theta)^\Delta {}_2F_1\left(\frac{\Delta+\ell+\omega}{2}, \frac{\Delta+\ell-\omega}{2}; \ell+\frac{d}{2}; \sin^2\theta\right) \tag{5}$$

$$G_- = (\sin\theta)^{(2-d-\ell)}(\cos\theta)^\Delta {}_2F_1\left(\frac{\Delta+2-d-\ell+\omega}{2}, \frac{\Delta+2-d-\ell-\omega}{2}; 2-\ell-\frac{d}{2}; \sin^2\theta\right) \tag{6}$$

$$\Delta \equiv \frac{d}{2} + \frac{1}{2}\sqrt{d^2+4m^2} \tag{7}$$

Hint: Use Mathematica.

Argue that asking the regularity at $\theta = 0$, $G'(\theta=0) = 0$ discards G_-.

(d) Consider the stress-energy tensor

$$T_{\mu\nu} = 2\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left((\partial\phi)^2 + m^2\phi^2\right) + \beta\left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + R_{\mu\nu}\right)\phi^2 \tag{8}$$

We require that the energy-momentum flux through the boundary ($\theta = \pi/2$) vanishes

$$\int_{S^{d-1}} d\Omega_{d-1}\sqrt{g}n_iT^i_0\Big|_{\theta=\pi/2} = 0 \tag{9}$$

Show that this requirement gives the boundary condition

$$(\tan\theta)^p[(1-2\beta)\partial_\theta + 2\beta\tan\theta]\phi^2 \to 0 \qquad (\theta\to\pi/2) \tag{10}$$

Argue that this is only satisfied when one of the two first arguments of G_+ is integer, implying

$$\pm\omega = \Delta + \ell + 2n \qquad (n=0,1,2,...) \tag{11}$$

(e) Since ω is interpreted as the energy, it must be real. Argue that this gives a lower bound on m^2, and give its value.

2. Lowest scalar energy state in AdS

Now consider AdS_{d+1} in global coordinates

$$ds^2 = -\cosh(\rho)^2 dt^2 + d\rho^2 + \sinh(\rho)^2 d\Omega_{d-1}^2 \tag{12}$$

(a) *(Optional)* Show that, in global coordinates of AdS_{d+1}, the conformal generators take the form

$$D = i\frac{\partial}{\partial t}, \qquad M_{\mu\nu} = -i\left(\Omega_\mu\frac{\partial}{\partial\Omega^\nu} - \Omega_\nu\frac{\partial}{\partial\Omega^\mu}\right),$$
$$P_\mu = -ie^{-it}\left[\Omega_\mu\left(\partial_\rho - i\tanh\rho\,\partial_t\right) + \frac{1}{\tanh\rho}\nabla_\mu\right],$$
$$K_\mu = ie^{it}\left[\Omega_\mu\left(-\partial_\rho - i\tanh\rho\,\partial_t\right) - \frac{1}{\tanh\rho}\nabla_\mu\right],$$

where

$$\nabla_\mu = \frac{\partial}{\partial\Omega^\mu} - \Omega_\mu\Omega^\nu\frac{\partial}{\partial\Omega^\nu}$$

is the covariant derivative on the unit sphere S^{d-1}.

(b) Impose that $K_\mu\phi = 0$, $D\phi = \Delta\phi$. Show that this restricts

$$\phi = \left(\frac{e^{-it}}{\cosh\rho}\right)^\Delta \tag{13}$$

(c) By changing from global to conformal coordinates, make connection with the previous exercise. Show in particular that this implies that Δ's are the same.

3. Vector fields in AdS

Consider the *Proca action* for a massive vector field in Lorentzian AdS in $D = d+1$ dimensions:

$$I = -\int d^Dx\sqrt{-g}\left(\frac{1}{4}g^{ac}g^{bd}F_{ab}F_{cd} + \frac{1}{2}m^2 g^{ab}A_aA_b\right), \tag{14}$$

where $F_{ab} \equiv \nabla_a A_b - \nabla_b A_a = \partial_a A_b - \partial_b A_a$.
In the Poincaré patch, we can write the metric as $g_{ab} = \Omega^2 \eta_{ab}$ with $\Omega = 1/z$, to get

$$I = -\int d^D x \left(\frac{1}{4} \Omega^{D-4} F_{ab} F^{ab} + \frac{1}{2} \Omega^{D-2} m^2 A_a A^a \right), \tag{15}$$

where indices are now raised with η_{ab}.

(a) How many physical degrees of freedom does a vector field contain when:
i) $m^2 > 0$, ii) $m^2 = 0$?

(b) Show that the equations of motion can be written as

$$\partial_a \left(z^{4-D} F^{ab} \right) = z^{2-D} m^2 A^b. \tag{16}$$

(c) Using the following ansatz for the asymptotic solution for the d components with $i \neq z$

$$A_i(x, z) = z^\nu J_i(x) + \mathcal{O}(z^{\nu+2}), \tag{17}$$

show that the equations of motion yield

$$\nu(\nu - 1) + (3 - d)\nu = m^2. \tag{18}$$

Consider the AdS dilatation isometry in Poincaré coordinates,

$$(x^i, z) \mapsto (x'^i, z') = (\lambda x^i, \lambda z). \tag{19}$$

Since $A = A_a\, dx^a$ is a bulk one-form, its boundary components transform under this diffeomorphism as

$$A'_i(x', z') = \frac{\partial x^j}{\partial x'^i} A_j(x, z) = \lambda^{-1} A_i(x, z). \tag{20}$$

We *define* Δ_J by requiring that J_i has boundary weight Δ_J under $x \mapsto \lambda x$,

$$J_i(\lambda x) = \lambda^{-\Delta_J} J_i(x), \tag{21}$$

(with the index transformation already accounted for by the one-form rule above).
By applying the dilatation to $A_i(x, z) = z^\nu J_i(x) + \cdots$, show that consistency implies

$$\Delta_J = \nu + 1 \tag{22}$$

(d) Now consider $m^2 = 0$. Show that we need

$$\Delta_J = d - 1 \quad \text{or} \quad \Delta_J = 1. \tag{23}$$

What are the physical interpretations of these two modes?
Hint: It might be helpful to think of which of the two modes decays the fastest, as well as the possible dual CFT operator.

(e) Since $m^2 = 0$, we have a gauge symmetry $\delta A_a = \nabla_a \alpha$, which can be fixed by imposing, for example, $A_z = 0$. Show that the z-equation of motion can be written as

$$\partial_i F^{iz} = 0 \tag{24}$$

Conclude from this that J_i can be interpreted as a conserved boundary current. In addition, check that the number of degrees of freedom of gauge vectors in D dimensions and conserved currents in d dimensions match.